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Interstate 69 and the Accessibility of Indiana's Major Cities

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Abstract: The political argument over whether to extend Interstate 69 through southwestern Indiana has been a long and difficult one. This article presents a mathematical model of the Indiana highway system, using weighted edges in a network and a weighted adjacency matrix. As explained in the article, the principal eigenvector of this adjacency matrix provides a method for rating how accessible the various cities are. In the model we "build" the proposed interstate highway and examine the resulting changes in accessibility.

Introduction

Since the late 1980s there has been a political debate in Indiana on whether a highway should be built through the southwest portion of the state, connecting the cities of Indianapolis, Bloomington, and Evansville. There is no interstate highway through this area, and proponents of the highway suggest that this has hindered the economic growth of these cities and of the rest of the region. They propose extending I-69 — which presently runs from Port Huron, Michigan to Indianapolis — as far as Evansville.

However, there are serious environmental issues involved. Nearly one hundred miles of new roadway would be required, passing through forests, wetlands, and park lands. The new highway would intersect prime farm land and limestone quarries. Any increase in traffic would exacerbate automobile emissions problems in Indianapolis and Evansville. The decision is thus a difficult one.

Changes in highway access clearly would affect the economies of the cities. Network theory and linear algebra provide one way to assess the influence of transportation links. Each city can be viewed as a node in a network and the highways as edges between nodes. An adjacency matrix records how the cities are connected. A certain eigenvector of this matrix then can be used to rate the "accessibility" of each node. (This method is discussed in Straffin [3], who used it to analyze villages and river ways in twelfth century Russia.) In our model of Indiana and its highway system, the proposed interstate highway can be built mathematically. This will give some indication of the changes it might cause. To improve the accuracy of the model, the connections will be weighted according to the type of highway involved. The volume of traffic is considerably different on different types of highway, and this can be reflected in the adjacency matrix.

The Network Model

The transportation system in Indiana is a collection of cities connected by transportation routes. These routes can be highways, railroads, airways, or waterways. We focus on highways and classify them into three types: the interstate, the four-lane, and the two-lane highways; see Figure 1. The accessibility of a city depends not only on the number of highways which run through it, but also on the classification of these highways. (This simplified map shows only the largest highways. In many instances there are parallel highways which must be included in our calculations.)

The cities and the highways connecting them can be viewed as a network of nodes and edges. This can be represented by an adjacency matrix, defined as

$$a_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are joined by an edge} \\ 0 & \text{if nodes } i \text{ and } j \text{ are not joined by an edge} \end{cases}$$

Further, we define $a_{ii} = 1$, so that each city is considered to be connected to itself. This will be important later when we interpret a matrix calculation.

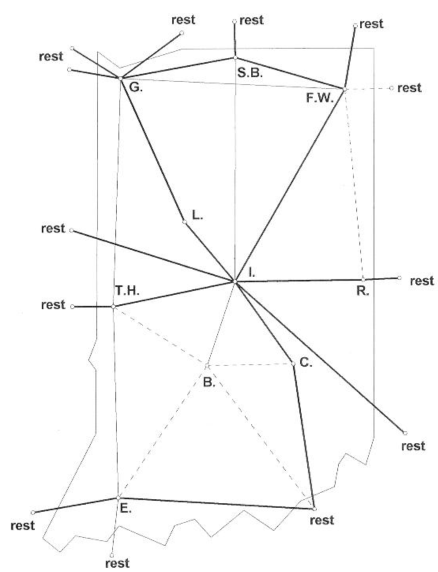

Figure 1: Major cities and highways in Indiana

For the state of Indiana we selected the ten major cities, as measured by total personal income. These are, in decreasing order: Indianapolis, Gary, Fort Wayne, Evansville, South Bend, Lafayette, Terre Haute, Bloomington, Columbus, and Richmond (United States Bureau of Economic Analysis [4]). Since most of these cities are connected to highways leaving the state, we added a node for "The Rest of the World". Here is the resulting adjacency matrix, which we call M:

$$
\begin{array}{ll}
 & \begin{array}{ccccccccccc} \text{B.} & \text{C.} & \text{E.} & \text{F.W.} & \text{G.} & \text{I.} & \text{L.} & \text{R.} & \text{S.B.} & \text{T.H.} & \text{Rest} \end{array} \\
\begin{array}{l} \text{Bloomington} \\ \text{Columbus} \\ \text{Evansville} \\ \text{Fort Wayne} \\ \text{Gary} \\ \text{Indianapolis} \\ \text{Lafayette} \\ \text{Richmond} \\ \text{South Bend} \\ \text{Terre Haute} \\ \text{The Rest} \end{array} &
\left(\begin{array}{ccccccccccc}
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

Notice the large number of connections to Indianapolis, which calls itself the "Crossroads of America". This suggests that Indianapolis should have a high accessibility rating. As we will see, this is indeed the case.

Thus far the three types of highway have not been reflected in the model. One way to account for the differences is to weight the edges according to the traffic flow on each type of highway. Using the annual daily traffic flow maps from the Indiana Department of Transportation [2], we found that the mean daily traffic flows were 36,435 vehicles traveling each Indiana interstate per day, 17,848 vehicles traveling each four-lane highway, and 10,612 vehicles traveling each two-lane highway. To our surprise, the traffic flow values were reasonably uniform across the state. Unfortunately, the data came county by county from different years in the span 1991–1995. So these means may not be completely reliable.

Rather than use the traffic flows as matrix entries, we used their ratios when compared to two-lane highways. These ratios are 3.433 for interstate highways, 1.682 for four-lane highways, and 1.000 for two-lane highways. This produced the following matrix W:

$$\begin{pmatrix}
1.682 & 1.000 & 1.000 & 0 & 0 & 1.682 & 0 & 0 & 0 & 1.000 & 0 \\
1.000 & 3.433 & 0 & 0 & 0 & 5.115 & 0 & 0 & 0 & 0 & 3.433 \\
1.000 & 0 & 3.433 & 0 & 0 & 0 & 0 & 0 & 0 & 1.682 & 8.548 \\
0 & 0 & 0 & 3.433 & 1.682 & 4.433 & 1.000 & 1.000 & 3.433 & 0 & 4.433 \\
0 & 0 & 0 & 1.682 & 3.433 & 0 & 5.115 & 0 & 3.433 & 1.682 & 10.299 \\
1.682 & 5.115 & 0 & 4.433 & 0 & 3.433 & 3.433 & 5.115 & 1.682 & 5.115 & 7.866 \\
0 & 0 & 0 & 1.000 & 5.115 & 3.433 & 3.433 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.000 & 0 & 5.115 & 0 & 3.433 & 0 & 0 & 4.433 \\
0 & 0 & 0 & 3.433 & 3.433 & 1.682 & 0 & 0 & 3.433 & 0 & 5.115 \\
1.000 & 0 & 1.682 & 0 & 1.682 & 5.115 & 0 & 0 & 0 & 3.433 & 4.433 \\
0 & 3.433 & 8.548 & 4.433 & 10.299 & 7.866 & 0 & 4.433 & 5.115 & 4.433 & 3.433
\end{pmatrix}$$

Here are two sample calculations. Consider the city of Evansville, which is the third row and the third column of W. From Evansville, Interstate 64 leaves Indiana both to the west and to the east. In addition, the four-lane highway U.S. 41 leaves the state to the south. Thus the total weight for the connection between Evansville and "The Rest of the World" — entry (3,11) in matrix W — is $2 \times 3.433 + 1.682$. For the connection between Indianapolis and Richmond, we must consider both Interstate 70 and the four-lane highway U.S.40. Thus the total weight is $3.433 + 1.682$.

The issue of entries on the diagonal is a difficult one. Each such entry represents how a city is connected to itself. By radically adjusting these values we were able to have the accessibility rankings occur in the same order as the total personal income rankings listed earlier. But we had no justification for the entries we used. Instead we chose to use a diagonal entry equal to the largest highway into that city. This did not produce exactly the order of income rankings, but there were many similarities. (This is discussed in more detail below.) The question of how to assign diagonal entries is unresolved.

The Perron-Frobenius Theorem

The accessibility ratings of the cities come from a particular eigenvector of the adjacency matrix. To explain which eigenvector to use and to justify its choice, we first need some mathematical background.

Notice that the weighted adjacency matrix W, like all adjacency matrices, has only real nonnegative entries and is symmetric. The spectral theorem for real symmetric matrices tells us that all eigenvalues of W are real and that the set of eigenvectors for this matrix W includes an orthogonal basis for $\Re^{11}$. Further, the matrix W is primitive, meaning that there is some exponent k for which all entries of W^k will be positive.

To see why W is primitive, we can use a theorem from graph theory. For a $0-1$ adjacency matrix with 1s on the diagonal, like our matrix M, the (i, j) entry of the power M^2 tells how many paths there are from node i to node j having length at most 2. (The length of a path is the number of edges the path uses.) Then the entries of M^3 count the number of paths of length at most 3, and so on. In our application, every pair of nodes (cities) has a path between them, so eventually a power of M will have only positive entries. A weighted edge can be interpreted as a conglomerate of individual edges, edges which are parallel in the network. For instance, the interstate between Indianapolis and Lafayette can be interpreted as 3.433 non-weighted edges. So the argument for matrix M still holds when we consider the matrix W.

This is what is needed to apply the following theorem. See Horn & Johnson [1] for a detailed discussion. (They use the term irreducible rather than primitive.)

Perron-Frobenius Theorem
A nonnegative primitive square matrix has an eigenvalue λ_1 which

- is real, positive, and is a simple root of the characteristic equation;
- is larger in magnitude than any other eigenvalue (so that $|\lambda_1| > |\lambda_i|$ when $\lambda_1 \neq \lambda_i$); and
- has a unique eigenvector $\vec{v_1}$, which can be chosen to have all positive entries and to have unit length.

We will refer to λ_1 as the principal eigenvalue, with $\vec{v_1}$ as the corresponding principal eigenvector. This is the eigenvector that provides the accessibility ratings of the ten cities.

To understand the connection between $\vec{v_1}$ and accessibility of nodes, we first make an important observation: for any vector $\vec{x}$ not orthogonal to $\vec{v_1}$, the product $W^k\vec{x}$ will approach a multiple of $\vec{v_1}$ as k increases. To see why this is true, begin by writing $\vec{x}$ as a linear combination of an orthogonal basis of eigenvectors of W. (The spectral theorem guarantees the existence of this basis.)

$$\vec{x} = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \cdots + \alpha_n\vec{v_n}$$

The coefficient α_1 cannot be 0, for if it were then the product $\vec{x}\cdot\vec{v_1}$ would also be 0 and the two vectors would be orthogonal. Form the product

$$W^k\vec{x} = \lambda_1^k\alpha_1\vec{v_1} + \lambda_2^k\alpha_2\vec{v_2} + \cdots + \lambda_n^k\alpha_n\vec{v_n}$$

Rearranging this gives

$$\frac{W^k\vec{x}}{\lambda_1^k} = \alpha_1\vec{v_1} + \left(\frac{\lambda_2}{\lambda_1}\right)^k\alpha_2\vec{v_2} + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k\alpha_n\vec{v_n}$$

Because λ_1 is the eigenvalue of greatest magnitude, the right hand side approaches $\alpha_1\vec{v_1}$ as $k \to \infty$. Consequently, the ratios of the components of $W^k\vec{x}$ will approach the ratios of the components of $\vec{v_1}$. Thus the product $W^k\vec{x}$ approaches a multiple of $\vec{v_1}$.

Why should the entries of the principal eigenvector tell us anything about accessibility of the nodes? There are at least two reasons to interpret these numbers in this way. One reason comes from the earlier argument showing that

W is primitive. Once a sufficiently large power W^k is computed, the (i, j) entry tells the total weighting of the paths between node i and node j. Adding the entries of row i tells the total weighting of paths leaving node i. A city with a greater total weight for its paths should be more accessible.

A convenient way to add the entries of the rows, and hence compute the total weight of paths for each node, is to calculate the product $W^k\vec{e}$, where $\vec{e}$ is the vector $(1, 1, \ldots, 1)$, containing entirely 1s. The vector $\vec{e}$ is entirely positive and so is the principal eigenvector $\vec{v_1}$, so the two vectors are not orthogonal. Therefore, by the theorem, $W^k\vec{e}$ approaches a multiple of $\vec{v_1}$, making the totals of these weights proportional to the entries of the principal eigenvector.

A second reason to interpret the entries of $\vec{v_1}$ as accessibility ratings comes from an analogy to the spread of a rumor. Imagine a rumor beginning with a person at node i at time 0. By time 1, that person tells people in each adjacent node and tells the rumor to people in his own city, with the number told dictated by the weighting. (Telling people in his own city is indicated by the entries on the diagonal.) By time 2, each person tells the rumor to additional people in adjacent nodes. As time progresses, which is represented by higher and higher powers of the matrix, the rumor spreads throughout the network. A node with more people who had heard the rumor would be rated as more accessible.

Again there is a convenient way to do the computations. Let $\vec{x_i}$ be the vector with 1 in the ith position and 0s elsewhere. This represents the start of the rumor at node i. The spread of the rumor is calculated by $W^k\vec{x_i}$ for increasing k. Since $\vec{x_i}$ is nonnegative, it is not orthogonal to $\vec{v_1}$. Thus the product converges to a multiple of the principal eigenvector, and gives the equilibrium distribution of the rumor.

Straffin [3] suggests that the rumor analogy also provides an interpretation for the principal eigenvalue, as the equilibrium growth rate of the rumor. A network with a larger λ_1 should allow a rumor to spread more quickly.

The Results

Computing eigenvalues and eigenvectors for an 11×11 matrix is a daunting task. Fortunately we were able to use the Maple software package to find approximate solutions to this problem. The approximate eigenvectors returned by Maple have unit length, so scaling is not an issue.

The first table below shows the accessibility rankings of these ten cities, without the proposed highway. The values shown are the components of the principal eigenvector. Comparing the two columns shows that the weightings of the highways have a strong effect on the rankings of the cities.

Table 1: Rankings from the Principal Eigenvector

Unweighted matrix M		Weighted matrix W	
$\lambda_1 = 5.7303$		$\lambda_1 = 27.2579$	
The Rest of the World	.4920	The Rest of the World	.5899
Indianapolis	.4413	Indianapolis (1)	.4226
South Bend	.3224	Gary (2)	.3583
Fort Wayne	.3215	Fort Wayne (3)	.2643
Terre Haute	.3145	South Bend (5)	.2462
Gary	.2705	Terre Haute (7)	.2443
Richmond	.2653	Evansville (4)	.2311
Columbus	.2212	Richmond (10)	.2116
Evansville	.1705	Columbus (9)	.1780
Lafayette	.1505	Lafayette (6)	.1489
Bloomington	.1132	Bloomington (8)	.0533

It is interesting that the rankings from the weighted matrix W agree somewhat with the total personal income rankings of these cities, shown in parentheses in Table 1. This suggests that the highway links are indeed major influences on the economies.

The next table shows the results of including the proposed new highway in our model. To "build" Interstate 69 required five changes in the matrix W: the connection from Indianapolis to Bloomington was changed to $1.682 + 3.433 = 5.115$, and vice-versa; the connection from Bloomington to Evansville was changed to $1.000 + 3.433 = 4.433$, and vice-versa; and the connection of Bloomington to itself was changed to 3.433.

Table 2: The Effects of Building the Highway

Without I-69		With I-69	
$\lambda_1 = 27.2579$		$\lambda_1 = 27.7457$	
The Rest of the World	.5899	The Rest of the World	.5766
Indianapolis	.4226	Indianapolis	.4348
Gary	.3583	Gary	.3424
Fort Wayne	.2643	Fort Wayne	.2558
South Bend	.2462	Evansville	.2477
Terre Haute	.2443	Terre Haute	.2438
Evansville	.2311	South Bend	.2359
Richmond	.2116	Richmond	.2071
Columbus	.1780	Columbus	.1792
Lafayette	.1489	Bloomington	.1540
Bloomington	.0533	Lafayette	.1439

The cities incident to the new highway — Indianapolis, Bloomington, and Evansville — all showed an increased value in the principal eigenvector. Given its two new major connections, it is not surprising that Bloomington had by far the greatest increase. All other cities had a decrease, except for Columbus, which had a slight increase. This may be a ripple effect from its proximity to both Indianapolis and Bloomington. There were only three changes in the rankings: Evansville passed both Terre Haute and South Bend into fourth position, Terre Haute and South Bend reversed positions, and Bloomington passed Lafayette into ninth position.

Conclusion

Our mathematical model supports the claim that building I-69 would have positive effects on the economies of the cities in southwest Indiana. But the model has weaknesses. First, only a few of the roadways in Indiana are included. Other less direct connections also carry traffic. Second, the weighting system relies on data collected in different years, which makes the mean traffic flow values somewhat suspect. A more accurate weighting would also reflect the variations in traffic flow across the state. Third, the weightings on the diagonal of the matrix need further analysis and justification. Fourth – and this is very important for an economic analysis – highway traffic is not the only mode of transportation in use. Railways and airways, even waterways, are significant influences, and electronic traffic is becoming a significant factor as well. Airways are particularly influential for the economies of the university towns: Bloomington, (West) Lafayette, and South Bend. An improved model would include these other modes of transportation and would have a more reliable weighting system. Perhaps the rankings then would correctly match the income rankings of the cities and allow a stronger conclusion.

There are additional questions in interpreting the results. Do the values in the principal eigenvector have a direct economic meaning? For instance, the accessibility rating for Bloomington almost tripled as a result of the new highway; would its economy triple? Would the increase in the principal eigenvalue represent an increased economy statewide? These questions deserve further investigation.

References

1. Roger A. Horn and Charles R. Johnson. Matrix Analysis. Cambridge University Press, 1985.

2. Indiana Department of Transportation, Division of Roadway Management. Annual Average Daily Traffic County Flow Maps. Indianapolis, Indiana, 1996.

3. Philip D. Straffin Jr., "Linear algebra in geography: eigenvectors of networks", Mathematics Magazine 53 n. 5 (November 1980), 269-276.

4. United States Bureau of Economic Analysis. Local Area Personal Incomes, 1969 - 1992.